SOME TORIC MANIFOLDS AND A PATH INTEGRAL

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1. Introduction

Toric manifolds and path integrals look like an odd couple at first sight, but are in fact intimately related, a phenomenon having its origin in statistical mechanics and quantum mechanics. It is not my intention to discuss this topic any in depth -and I am ill equipped to do so in any event- but only to suggest that this curiosity deserves study. It is in this spirit that I have mixed some physical terminology into what is really a mathematical exercise.

The toric manifolds in question are of a very special type. They were invented by Bott and studied by Grossberg and Karshon [1994]. Their intriguing results stimulated my interest in these matters, especially in the symplectic geometry of these manifolds. One thing that is new here is the construction of 2-forms on these manifolds as curvature forms of holomorphic line bundles with unitary metrics. These 2- forms are the objects of interest and their realization here leads naturally to an explicit formula from which everything else follows by simple calculations, things like canonical coordinates and positivity criteria. The more important innovation, however, seems to me the method, using a construction of these toric manifolds as a special case of what I call '$\mathbb{P}^1$-chains', a construct for which I have some hope. The toric example was worked out as a test for the general case, to be discussed elsewhere. Here the methods are completely elementary, relying on direct calculations presented in calculus style.

A major reason for studying these manifolds comes from Kirillov's [1968] orbit theory for unitary representations of Lie groups and Kostant's [1970] theory of quantization. The point of view here is to look for the character of a unitary representation rather than for the index of a virtual representation. This requires some vanishing theorems for cohomology, which I deduce from a positivity result for the curvature by Kodaira's theorem. The unitary structure on the representation space depends on the curvature form and its positivity as well and the spectrum of the representation is determined with the help of the Kähler potential and its convexity.

Kirillov's universal formula for the characters of representations constructed by quantization has inspired much of what I have to say on this topic here. In keeping with the physical origins of the subject, I have taken the opportunity to explain in some detail two formulas for the character in question -alias partition function- one along the lines of statistical mechanics, the other in terms of a path integral. The first one is very simple, but seems of some merit to me; the second one, the path integral formula, seems of some interest as well, even though it may be commonplace in a formal way. The point is simply to write out rather carefully

how the path integral can be defined in the situation at hand. The proof that it does give the right answer is then immediate in any case, although the reason why it does so -and not only here but also in many other situations- is far from clear.

The paper consists then really of two parts, one dealing with the very special toric varieties mentioned, the other with rather more general questions, for which these toric varieties serve as vehicle of exposition. It seemed to me that each part benefits from the other.

2. $\mathbb{P}^1$ ITSELF

Write G for $\mathrm{SL}(2,\mathbb{C})$ acting by holomorphic automorphisms on $\mathbb{P}^1$,

$$z(g\xi) = \frac{az(\xi)+c}{bz(\xi)+d}, \quad ad-bc=1.$$

The points ξ_0, ξ_∞ with $z=0,\infty$ are the fixed points of the subgroup T of diagonal matrices and $z=z(\xi)\in\mathbb{C}\cup\{\infty\}$ is the affine coordinate on $\mathbb{P}^1$ centered at ξ_0. The weights by which T acts on the cotangent spaces at these points ξ_0,ξ_∞, denoted $\alpha,-\alpha$, are twice the weights on the two coordinate lines $\mathbb{C}e_{\alpha/2}, \mathbb{C}e_{-\alpha/2}$ in $\mathbb{C}^2$ which correspond to the points ξ_0,ξ_∞ in $\mathbb{P}^1$. They are interchanged by the reflection $s=s_\alpha$, transforming the affine coordinate z centered at ξ_0 into the affine coordinate $z\circ s = -1/z$ centered at ξ_∞. Choose ξ_0 as base-point for $\mathbb{P}^1$ and let B denote its isotropy group in G, thus providing the realization of $\mathbb{P}^1$ as the coset space G/B. The triple (G, B, T) determines uniquely a basis $H_\alpha, E_\alpha, E_{-\alpha}$ of the Lie algebra of G so that T is generated by H_α, B by H_α together with E_α, and

$$[H_\alpha, E_{\pm\alpha}] = \pm 2E_{\pm\alpha}, \quad [E_\alpha, E_{-\alpha}] = H_\alpha.$$

Write N and N^- for the subgroups generated by E_α and $E_{-\alpha}$, respectively. The affine coordinate $z=z(g\mathrm{B})$ on G/B centered at ξ_0 is determined by $v=e^{z(g)E_{-\alpha}}$ if $g=vcn$ according to $\mathrm{G}\doteq\mathrm{N}^-\mathrm{TN}$. The symbol $\doteq$ indicates that the decomposition is valid only for $g\mathrm{B}\neq\xi_\infty$. The compact real form U :=SU(2) of G has $\mathrm{i}H_\alpha, E_\alpha\pm\mathrm{i}E_{-\alpha}$ as basis for its Lie algebra.

Homomorphisms $\mathrm{T}\to\mathrm{GL}(1,\mathbb{C})$ will be written exponentially as $h\mapsto h^\lambda$, λ being a linear functional on the Lie algebra, referred to as a weight of T. A homomorphism of T extends to B = TN so that $b^\lambda = h^\lambda$ if $b=hn$. $\mathbb{C}^2$ carries a symplectic form invariant under G and a unitary metric invariant under U. Write $g\mapsto g^*$ for the involution of G acting as $u^*=u^{-1}$ on U. The decomposition G= UB, $g=ub$, is valid everywhere and unique if $b\in\mathrm{B}=\mathrm{TN}$ is normalized so that its component in T is real and positive.

The complex-valued holomorphic functions $f(g)$ on G satisfying $f(gb)=f(g)b^\lambda$ are the holomorphic sections of a line bundle $\mathcal{L}_\lambda$ on $\mathbb{P}^1=\mathrm{G/B}$. In this capacity $f(\xi)$ stands for the element in the line $\mathcal{L}_\lambda(\xi)$ at $\xi=g\mathrm{B}$. This line bundle has a U invariant unitary metric given by

$$|f(\xi)|^2 := |f(u)|^2 \quad \text{if } \xi=u\mathrm{B},\ u\in\mathrm{U}.$$

One such function is $f(g) := \langle e_{-\alpha/2}, ge_{\alpha/2}\rangle^l$, the pointed brackets denoting the G-invariant symplectic pairing on $\mathbb{C}^2$. The curvature form σ_λ of $\mathcal{L}_\lambda$, normalized so that it represents the Chern class, is

$$\sigma_\lambda = \frac{-\lambda(H_\alpha)}{2\pi\mathrm{i}}\frac{d\overline{z}\wedge dz}{(1+\overline{z}z)^2}\,.$$

This $(1,1)$-form σ_λ is *negative* when $\lambda(H_\alpha)$ is positive. These matters will be explained in a more general setting when needed.

In short, the triple $(\mathrm{G},\mathrm{B},\mathrm{T})$ determines the basis $(H_\alpha, E_\alpha, E_{-\alpha})$, which allows the reconstruction of $\mathbb{P}^1$, complete with its affine coordinates z, the matrix representation of its automorphism group on $\mathbb{C}^2$, its line bundles $\mathcal{L}_\lambda$ with their unitary structures, etc. All of these $\mathbb{P}^1$-paraphernalia are thus available for any triple of groups known to be abstractly isomorphic with $(\mathrm{G},\mathrm{B},\mathrm{T})$. Nevertheless, for the purpose of calculation in a more general setting it is preferable to work directly with the triple $(\mathrm{G},\mathrm{B},\mathrm{T})$. Three of its properties are assembled here for reference.

Let $v(g), c(g)$ be the components in N^-, T according to the decomposition $\mathrm{G} \doteq \mathrm{N}^-\mathrm{TN}$. Let $u(g), b(g)$ be components in U, B according to G= UB, normalized so that $b(g) \in a(g)\mathrm{N}$ with $a(g) \in \mathrm{T}$ real and positive.

Lemma 2.1. *If $l := \lambda(H_\alpha) = 0, 1, 2, \cdots$ is a non-negative integer, then $c_s^\lambda(g) := c(sg)^\lambda$ extends to a holomorphic function on all of* G and $c_s^\lambda(g) = 0$ iff $g \in s\mathrm{B}$.

Proof. In terms of the G-invariant pairing on $\mathbb{C}^2$,

$$c_s^{\alpha/2}(g) = \langle se_{\alpha/2}, ge_{\alpha/2}\rangle, \quad \langle se_{\alpha/2}, e_{\alpha/2}\rangle = 1.$$

If $l := \lambda(H_\alpha)$ then $\lambda = l(\alpha/2)$ and

$$c_s^\lambda(g) = \langle se_{\alpha/2}, ge_{\alpha/2}\rangle^l.$$

The assertion follows. □

Lemma 2.2. *If $u = u(g)$, $v = v(g)$ for some g, then $c(u) = c^{-1}(b(v))$.*

Proof. If $u = u(g)$, $v = v(g)$ then $u\mathrm{B} = v\mathrm{B}$ so $v = ub$ for some b necessarily equal to $b(v)$. Hence $u = vb^{-1}$ and $c(u) = c(b^{-1}) = c^{-1}(b)$. □

The third lemma is the essential one; its formulation may seem awkward at this point.

Lemma 2.3. *Let $e^{zE_{-\alpha}} \in \mathrm{N}^-$, $z = re^{\mathrm{i}\phi}$, and $a \in \mathrm{T}$ real and positive. Then*

$$b(ae^{zE_{-\alpha}}) \in \tilde{a}\mathrm{N}, \quad \tilde{a} := a\mathrm{a}^{H_\alpha/2}(\tilde{r}), \quad \tilde{r} := a^{-\alpha}r$$

where

$$\mathrm{a}(r) := 1 + r^2.$$

and $\mathrm{a}^{H_\alpha/2} := e^{\frac{1}{2}(\log \mathrm{a})H_\alpha}$.

Proof. First take $a = 1$ and set $v = e^{zE_{-\alpha}}$. Then $ve_{\alpha/2} = e_{\alpha/2} + ze_{\alpha/2}$. Write $v = ub$ and $b \in \tilde{a}\mathrm{N}$ with $\tilde{a}$ real and positive. Then $ve_{\alpha/2} = \tilde{a}^{\alpha/2}ue_{\alpha/2}$. Hence

$$\tilde{a}^{\alpha/2}ue_{\alpha/2} = e_{\alpha/2} + ze_{\alpha/2}$$

Taking square norms and using the normalization $\tilde{a} =$real, positive, find

$$\tilde{a}^\alpha = 1 + r^2 = \mathrm{a}(r).$$

Hence $\tilde{a} = \mathrm{a}^{-H_\alpha/2}(z)$, which is the desired relation $b(e^{zE_{-\alpha}}) \in \mathrm{a}^{H_\alpha/2}(z)\mathrm{N}$ for $a = 1$. For general a,

$$b(ae^{zE_{-\alpha}}) = b(e^{a^{-\alpha}zE_{-\alpha}})a \in \mathrm{a}^{H_\alpha/2}(a^{-\alpha}z)a\mathrm{N}\,.$$

□

3. Definitions

Let $(G_\ell, B_\ell, T_\ell), \cdots (G_1, B_1, T_1)$ be a sequence obtained from standard $\mathbb{P}^1$-triples $(\mathrm{G}_{\alpha_i}, \mathrm{B}_{\alpha_i}, \mathrm{T}_{\alpha_i})$ by simply enlarging the 1-tori T_i to bigger tori T_i through central extensions:

$$G_i = \mathrm{G}_{\alpha_i} T_i, \quad B_i = \mathrm{B}_{\alpha_i} T_i, \quad \mathrm{G}_{\alpha_i} \cap T_i = \mathrm{T}_{\alpha_i} = \mathrm{B}_{\alpha_i} \cap T_i\,.$$

All groups are required to be complex algebraic. The scheme is to denote by roman letters objects belonging to $\mathbb{P}^1$ itself, like the triples $(\mathrm{G}_\alpha, \mathrm{B}_\alpha, \mathrm{T}_\alpha)$. In addition, let $T_i \to T_{i-1}, h_i \mapsto h_i \gamma_{i\,i-1}$, be a sequence of homomorphisms of these tori, written on the *right*. The semidirect decomposition $\mathrm{B}_{\alpha_i} = \mathrm{T}_{\alpha_i} \mathrm{N}_{\alpha_i}$ extends to $B_i = T_i \mathrm{N}_{\alpha_i}$ and the homomorphisms extend to

$$B_i = \mathrm{N}_{\alpha_i} T_i \to T_i \stackrel{\gamma_{i\,i-1}}{\longrightarrow} T_{i-1} \subset B_{i-1},$$

producing the sequence

$$G_\ell \stackrel{\subseteq}{\leftarrow} B_\ell \stackrel{\gamma_{\ell,\ell-1}}{\to} G_{\ell-1} \stackrel{\subseteq}{\leftarrow} \cdots \stackrel{\subseteq}{\leftarrow} B_2 \stackrel{\gamma_{21}}{\to} G_1\,.$$

The $\mathbb{P}^1$-*chain* associated to this sequence is the quotient

$$\mathcal{X} := G_\ell \times_{B_\ell} \times G_{\ell-1} \times \cdots \times_{B_2} \times G_1/B_1$$

of $G_\ell \times G_{\ell-1} \times \cdots \times G_1$ by the (free) right action of $B_\ell \times B_{\ell-1} \times \cdots \times B_1$ defined by

$$(p_\ell, p_{\ell-1}, \cdots, p_1) \mapsto (p_\ell b_\ell, b_\ell^{-1} \gamma_{\ell\,\ell-1} p_{\ell-1}, \cdots p_2 b_2, b_2^{-1} \gamma_{21} p_1).$$

The homomorphisms γ are the *connecting homomorphisms* of the $\mathbb{P}^1$-chain. The construction can proceed one step at a time, from right to left. Write $\mathrm{s} := (G_\ell, B_\ell, T_\ell, \gamma_{\ell\ell-1}, \cdots, \gamma_{21}, G_1, B_1, T_1)$ for the whole sequence of data which goes into the construction and $\mathcal{X} = \mathcal{X}_\mathrm{s}$ for the result. Write ξ_0 for the element of $\mathcal{X}_\mathrm{s}$ represented by $g_i = 1$ for all i, $\xi = g_\ell . g_{\ell-1} . \cdots . g_1 . \xi_0$ for the element represented by $(g_\ell, g_{\ell-1}, \cdots, g_1)$ and $G_\ell . G_{\ell-1} . \cdots . G_1 . \xi_0$ for $\mathcal{X}_\mathrm{s}$ itself. For brevity, write also ξ as $g_{\ell-1} . g_\ell \cdots$ and $\mathcal{X}_\mathrm{s}$ as $G_\ell . G_{\ell-1} \cdots$. Equality in the quotient $\mathcal{X}_\mathrm{s}$ means

$$g_\ell b_\ell . g_{\ell-1} \cdots = g_\ell . b_\ell \gamma_{\ell\,\ell-1} g_{\ell-1} \cdots\,.$$

Omit $\gamma_{\ell\,\ell-1}$ and write $b_\ell g_{\ell-1}$ for $b_\ell \gamma_{\ell\,\ell-1} g_{\ell-1}$ when the meaning is clear from the indices. Let $G_\mathrm{s} := \prod_\mathrm{s} g_i, B_\mathrm{s} := \prod_\mathrm{s} B_i$ denote the direct products and $\mathcal{X}_\mathrm{s} := G_\mathrm{s}/_\gamma B_\mathrm{s}$, the quotient by the right action of B_s on G_s defined above. The product torus $T_\mathrm{s} := \prod_\mathrm{s} T_i$ acts on $\mathcal{X}_\mathrm{s}$ on the left, denoted $\xi \mapsto h\xi$ and given by

$$g_\ell . g_{\ell-1} . \cdots \mapsto h_\ell g_\ell . h_{\ell-1} g_{\ell-1} . \cdots\,.$$

T_s has an open dense orbit on $\mathcal{X}_\mathrm{s}$, so that $(\mathcal{X}_\mathrm{s}, T_\mathrm{s})$ is a smooth toric variety in the usual sense.

Associated to the sequence $(G_i, B_i, T_i, \gamma_{ii-1})$ is the sequence of roots α_i and a system of integers $c_{ji}, j > i$, defined as follows. Let $h_j \to h_j \gamma_{ji}$ be the composite of the connecting homomorphisms from T_j to T_i and let $\gamma_{ji} \alpha_i \leftarrow \alpha_i$ be the pull-back of the root to T_j from T_i . The value of $\gamma_{ji} \alpha_i$ on the canonical generator H_{α_j} of the subtorus T_{α_j} of T_j is an integer denoted $c_{ji} := \langle H_{\alpha_j}, \gamma_{ji} \alpha_i \rangle$. The homomorphism γ_{ji} will be omitted if the meaning of $\langle H_{\alpha_j}, \alpha_i \rangle$ is clear from the indices.

Some comments. The example which motivated the definition is the case when the (G_i, B_i, T_i) are triples naturally associated to a sequence of roots α_i for a triple (G, B, T) consisting of a semisimple group G, a Borel subgroup B and a Cartan subgroup T. In this case $T_i = T$ for all i and one can take the identity map

$T \to T$ to construct the connecting homomorphisms $B_i \to B_{i-1}$. If one replaces the triples (G_i, B_i, T) by triples (P_i, B, T) where the P_i are minimal (proper) parabolic subgroups of G containing B and takes the identity $B \to B$ for connecting homomorphisms one obtains the well-known Bott-Samelson manifolds. Other connecting homomorphisms, which are generally an essential ingredient, are of interest as well, e.g. those induced by conjugations in G for the triples (P_i, B, T). Manifolds isomorphic to the toric $\mathbb{P}^1$-chains defined above were first defined by Bott using a different procedure and studied by Grossberg and Karshon [1994], as mentioned in the introduction. Although it would not be worth while introducing $\mathbb{P}^1$-chains constructed by means of connecting homomorphisms just for the case discussed here, there is a generalization for which it is: each triple (G_i, B_i, T_i) can be taken to consist of an arbitrary algebraic groups G_i whose quotient G_i/B_i by its maximal solvable subgroup is isomorphic with $\mathbb{P}^1$. An extension of $\mathrm{SL}(2,\mathbb{C})$ by a Heisenberg-Weyl group is an example, beyond those already mentioned.

4. The curvature formula

Let $\mathcal{X}_{\mathrm{s}}$ be the $\mathbb{P}^1$-chain built from a sequence

$$\mathrm{s} := (G_\ell, B_\ell, T_\ell\,, \gamma_{\ell\,\ell-1}, \cdots \gamma_{21}, G_1, B_1, T_1\,).$$

Line bundles. Let $c(g_i)$ be the component in T_i according to the decomposition $G_i \doteq \mathrm{N}^-_{\alpha_i} T_i N_{_i}$, defined for g_i in this open set. A weight λ of the product torus T_{s} is a sequence $\lambda = (\lambda_i)$. Define a function c^λ_{s} on the open set $\prod_{\mathrm{s}} B_i s_i B_i$ in G_{s} by the formula

$$c^\lambda_{\mathrm{s}}(g_{_\ell}, \cdots, g_1) := \prod_{\mathrm{s}} c^{\lambda_i}_{s_i}(g_i), \quad c_{s_i}(g_i) := c(s_i g_i).$$

This function is defined on the domain where all the factors are defined, i.e. $g_i \in s_i N^-_i B_i = B_i s_i B_i$. We record a transformation property of the function c^λ_{s} under the right action of B_{s} on G_{s}. Let $\gamma_{i\,i-1}\lambda_{i-1}$ denote the pull-back of a weight λ_{i-1} from B_{i-1} toB_i and set

$$\gamma\lambda := (\gamma_{\ell\,\ell-1}\lambda_{\ell-1}, \cdots \gamma_{21}\lambda_1, 0), \quad \mathrm{s}\lambda := (s_\ell\lambda_\ell, \cdots, s_1\lambda_1).$$

Also set $w_i := s_i \cdots s_1$ and $\mathrm{w}\lambda := (w_\ell\lambda_\ell, \cdots, w_1\lambda_1)$.

Lemma 4.1. *As function of $g \in G_{\mathrm{s}}$, $c^\lambda_{\mathrm{s}}(g)$ transforms under the right action of b according to the rule $c^\lambda_{\mathrm{s}}(g \cdot b) = c^\lambda_{\mathrm{s}}(g) b^{\tilde\lambda}$ where*

$$\tilde\lambda := (1 - \mathrm{s}\gamma)\lambda,\ \lambda = (1 + \mathrm{w}\,\gamma)\tilde\lambda.$$

Proof. Omit the connecting homomorphisms from the notation to calculate

$$\begin{aligned} c^\lambda_{\mathrm{s}}(g \cdot b) &= c^\lambda_{\mathrm{s}}(g_\ell b_{\ell-1}, b_\ell^{-1} g_{\ell-1} b_{\ell-1}, \cdots b_2^{-1} g_1 b_1) \\ &= c^{\lambda_\ell}(s_\ell g_\ell b_\ell)\, c^{\lambda_{\ell-1}}(s_{\ell-1} b_\ell^{-1} g_{\ell-1} b_{\ell-1}) \cdots c^{\lambda_1}(s_1 b_2^{-1} g_1 b_1) \\ &= c^{\lambda_\ell}(s_\ell g_\ell) b_\ell^{\lambda_{\ell-1}} b_\ell^{-s_{\ell-1}\lambda_{\ell-1}} c^{\lambda_{\ell-1}}(g_{\ell-1}) b_{\ell-1}^{\lambda_{\ell-1}} \cdots b_2^{-s_1\lambda_1} c^{\lambda_1}(g_1) b_1^{\lambda_1} \\ &= c^\lambda_{\mathrm{s}}(g) b^{\tilde\lambda} \end{aligned}$$

One checks that $\tilde\lambda := (1 - \mathrm{s}\gamma)\lambda$ is equivalent to $\lambda = (1 + \mathrm{w}\,\gamma)\tilde\lambda$. ∎

The locally defined holomorphic functions $f(g)$ on G_{s} satisfying $f(g\cdot b)=f(g)b^{\tilde\lambda}$ where defined are the local sections of a holomorphic line bundle $\mathcal{L}_\lambda$ on $\mathcal{X}_{\mathrm{s}}$. Write $f(\xi)$ for the value of the section f in the line $\mathcal{L}_\lambda(\xi)\approx\mathbb{C}$ attached $\xi\in\mathcal{X}_{\mathrm{s}}$. This line $\mathcal{L}_\lambda(\xi)$ carries a unitary metric defined by $|f(\xi)|^2:=|f(u)|$ if $\xi=u.\xi_0$ with $u\in U_{\mathrm{s}}:=\prod_{\mathrm{s}}U_i$, U_i being the compact form of G_i. To make use of fiber bundle technology is useful to note this reformulation of the lemma.

Corollary 4.2. *$\mathcal{L}_\lambda=G_{\mathrm{s}}\times_{B_{\mathrm{s}}}\mathbb{C}_{\tilde\lambda}$ is the holomorphic line bundle on $\mathcal{X}_{\mathrm{s}}=G_{\mathrm{s}}/_\gamma B_{\mathrm{s}}$ associated to the 1-dimensional representation $\tilde\lambda=(1-\mathrm{s}\gamma)\lambda$ of the structure group B_{s} of the principal bundle $G_{\mathrm{s}}\to\mathcal{X}_{\mathrm{s}}$. On the underlying real manifold $\mathcal{X}_{\mathrm{s}}=U_{\mathrm{s}}/_\gamma T_{\mathrm{sU}}$ ($T_{\mathrm{sU}}:=T_{\mathrm{s}}\cap U_{\mathrm{s}}$) the line bundle is $\mathcal{L}_\lambda=U_{\mathrm{s}}\times_{T_{\mathrm{sU}}}\mathbb{C}_{\tilde\lambda}$ and its unitary metric is associated to the T_{sU} invariant unitary metric on $\mathbb{C}_{\tilde\lambda}$.* □

Any holomorphic line bundle $\mathcal{L}$ equipped with a unitary structure has a $(1,1)$-form σ as curvature. Normalized so that it represents the Chern class, this form is given by

$$\sigma=-\frac{1}{2\pi\mathrm{i}}\bar\partial\partial\log|f|^2$$

for any local holomorphic section f, the formula being valid where $f\neq 0$ [Giffiths and Harris, 1978, p.142]. It will be computed explicitly for the case at hand.

Affine coordinates. The equation

$$\xi:=e^{z_\ell E_{\alpha_\ell}}s_\ell.e^{z_{\ell-1}E_{\alpha_{\ell-1}}}.s_{\ell-1}\cdots e^{z_1E_{\alpha_1}}s_1.\xi_0.$$

defines holomorphic functions $z_\ell,z_{\ell-1},\cdots,z_1$ on the image of the dense open set $\prod_{\mathrm{s}}B_is_iB_i$ under the quotient map $G_{\mathrm{s}}\to\mathcal{X}_{\mathrm{s}}$. Call $(z_\ell,\cdots,z_1)$ the *affine coordinates* on $\mathcal{X}_{\mathrm{s}}$ centered at $\xi_\infty:=s_\ell.s_{\ell-1}\cdots s_1.\xi_0$. These functions $z_i=z_i(\xi)$ are well-defined because $G_{\mathrm{s}}\to\mathcal{X}_{\mathrm{s}}$ maps $\prod_{\mathrm{s}}\mathrm{N}_is_i$ isomorphically onto its image, but their definition requires that ξ be written in the form indicated.

Lemma 4.3. *Under the action of an element $h=(h_i)$ of T_{s} the affine coordinates (z_i) transform according to the rule*

$$z_i\circ h=h^{\varpi_i}z_i,\quad \varpi_i:=(w_{\ell i}\alpha_i,\cdots,w_{i+1\,i}\alpha_i).$$

$w_{ji}:=s_j\gamma_{j\,j-1}s_{j-1}\cdots\gamma_{i+1i}s_i$ is the homomorphism $T_j\to T_i$ composed of the reflections s and the connecting homomorphisms γ.

Proof. Let $\xi:=\exp(z_\ell E_{\alpha_\ell})s_\ell.\exp(z_{\ell-1}E_{\alpha_{\ell-1}}).s_{\ell-1}\cdots$. Omit the connecting homomorphisms from the notation to calculate

$$\begin{aligned}h\xi&:=h_\ell\exp(z_\ell E_{\alpha_\ell})s_\ell.h_{\ell-1}\exp(z_{\ell-1}E_{\alpha_{\ell-1}}).s_{\ell-1}\cdots\\&=\exp(h_\ell^{\alpha_\ell}z_\ell E_{\alpha_\ell})h_\ell s_\ell.h_{\ell-1}\exp(z_{\ell-1}E_{\alpha_{\ell-1}}).s_{\ell-1}\cdots\\&=\exp(h_\ell^{\alpha_\ell}z_\ell E_{\alpha_\ell})s_\ell.h_\ell^{s_\ell}h_{\ell-1}\exp(z_{\ell-1}E_{\alpha_{\ell-1}}).s_{\ell-1}\cdots\\&=\exp(h_\ell^{\alpha_\ell}z_\ell E_{\alpha_\ell})s_\ell.\exp(h_\ell^{s_\ell\alpha_{\ell-1}}h_{\ell-1}^{\alpha_{\ell-1}}z_{\ell-1}E_{\alpha_{\ell-1}})h_\ell^{s_\ell}h_{\ell-1}.s_{\ell-1}\cdots\end{aligned}$$

etc. This gives

$$z_i\circ h=(h_\ell^{w_{\ell i}\alpha_i}h_{\ell-1}^{w_{\,\ell-1i}\alpha_i}\cdots h_{i+1}^{w_{\,i+1\,i}\alpha_i}h_i^{\alpha_i})z_i,$$

and hence the assertion. □

The points in the coordinate domain with all coordinates non-zero form a single orbit of T_{s}, namely its open dense orbit on the toric manifold $\mathcal{X}_{\mathrm{s}}$.

Curvature. Let $\lambda=(\lambda_i)$ be a weight of T_{s} and $\mathcal{L}_\lambda$ the associated holomorphic line bundle on $\mathcal{X}_{\mathrm{s}}$ with its unitary metric. The following formula gives its curvature.

Theorem 4.4. *The curvature form σ_λ of $\mathcal{L}_\lambda$ is given by*

$$\sigma_\lambda = +\frac{1}{2\pi \mathrm{i}} \sum_{ij} \frac{\partial^2 K_\lambda}{\partial \bar{z}_i \partial z_j} d\bar{z}_i \wedge dz_j\,, \quad K_\lambda := \log a_\lambda,$$

$$a_\lambda := \mathrm{a}(\tilde{r}_1)^{l_1} \cdots \mathrm{a}(\tilde{r}_\ell)^{l_\ell}, \quad l_i := \lambda_i(H_{\alpha_i}), \quad \mathrm{a}(\tilde{r}) := 1 + \tilde{r}^2.$$

$\tilde{r}_\ell, \cdots, \tilde{r}_1$ are the functions of $r_\ell, \cdots, r_1$ and v.v. defined inductively by

$$r_\ell = \tilde{r}_\ell, \quad r_i = \tilde{r}_i \prod_{j:j>i} \mathrm{a}^{c_{ji}/2}(\tilde{r}_j).$$

The exponents c_{ji} are the integers $c_{ji} = \langle H_{\alpha_j}, \alpha_i \rangle$.

The variables $\tilde{r}_i$ are defined where the $z_i = r_i e^{\sqrt{-1}\phi_i}$ are and $(\tilde{r}_i,\, \phi_i)$ may be used as real coordinates on $\mathcal{X}_\mathrm{s}$ instead of (r_i, ϕ_i). Their recursive definition proceeds in the direction opposite to the construction of the $\mathbb{P}^1$- chain. Combine the $(\tilde{r}_i,\, \phi_i)$ into complex valued variables $\tilde{z}_i := \tilde{r}_i e^{\sqrt{-1}\tilde{\phi}_i}$, considered as functions of the $z_i = r_i e^{\sqrt{-1}\phi_i}$ by the above equations supplemented by $\phi_i = \tilde{\phi}_i$, even though these functions $\tilde{z}_i$ are not holomorphic. The angle variables $\phi_i = \tilde{\phi}_i$ play no role in much of what is to follow and will then be suppressed. (The hoary notion of 'variables', which leaves it to reader and context to decide what is a function of what and in what way, is most convenient here and even more so later.)

Proof. To simplify the notation take a $\mathbb{P}^1$-chain of length three, and take (α, β, γ) as indices. Use the shorthand notation like $\xi = g_\alpha . g_\beta . g_\gamma . \xi_0$ etc, and omit the connecting homomorphisms. By definition

$$|c^\lambda_\mathrm{s}(g_\alpha . g_\beta . g_\gamma . \xi_0)|^2 = |c^{\lambda_\alpha}(u_\alpha)|^2 |c^{\lambda_\beta}(u_\beta)|^2 |c^{\lambda_\gamma}(u_\gamma)|^2,$$

if $g_\alpha . g_\beta . g_\gamma . \xi_0 = u_\alpha . u_\beta . u_\gamma . \xi_0$. In terms of the components $u(g) \in \mathrm{U}, b(g) \in \mathrm{B}$ according to $\mathrm{G} = \mathrm{UB}$,

$$\begin{aligned}
& s_\alpha g_\alpha . s_\beta g_\beta . s_\gamma g_\gamma . \xi_0 = \\
&= u_\alpha b_\alpha . s_\beta g_\beta . s_\gamma g_\gamma . \xi_0 \quad [u_\alpha := u(s_\alpha g_\alpha), b_\alpha := b(s_\alpha g_\alpha)] \\
&= u_\alpha . u_\beta b_\beta . s_\gamma g_\gamma . \xi_0 \quad [u_\beta := u(b_\alpha s_\beta g_\beta), b_\beta := b(b_\alpha s_\beta g_\beta)] \\
&= u_\alpha . u_\beta . u_\gamma b_\gamma . \xi_0 \quad [u_\gamma := u(b_\beta s_\gamma g_\gamma), b_\gamma := b(b_\beta s_\gamma g_\gamma)] \\
&= u_\alpha . u_\beta . u_\gamma . \xi_0
\end{aligned}$$

In terms of the components $v(g) \in \mathrm{N}^-$ according to $\mathrm{G} \doteq \mathrm{N}^-\mathrm{TN}$ (Lemma 2.2, $b^\lambda := c(b)^\lambda$),

$$\begin{aligned}
|c^{\lambda_\alpha}(u_\alpha)|^2 &= b^{-2\lambda_\alpha}(\tilde{v}_\alpha) \quad [\tilde{v}_\alpha := v(s_\alpha g_\alpha)] \\
|c^{\lambda_\beta}(u_\beta)|^2 &= b^{-2\lambda_\beta}(\tilde{v}_\beta) \quad [\tilde{v}_\beta := v(b_\alpha s_\beta g_\beta)] \\
|c^{\lambda_\gamma}(u_\gamma)|^2 &= b^{-2\lambda_\gamma}(\tilde{v}_\gamma) \quad [\tilde{v}_\gamma := v(b_\beta s_\gamma g_\gamma)].
\end{aligned}$$

Define $\tilde{z}_\alpha, \tilde{z}_\beta, \tilde{z}_\gamma$ by

$$e^{\tilde{z}_\alpha E_{-\alpha}} := \tilde{v}_\alpha, e^{\tilde{z}_\beta E_{-\beta}} := \tilde{v}_\beta, e^{\tilde{z}_\gamma E_{-\gamma}} := \tilde{v}_\gamma.$$

so that

$$|c^\lambda_\mathrm{s}(g_\alpha . g_\beta . g_\gamma . \xi_0)|^2 = b^{-2\lambda_\alpha}(e^{\tilde{z}_\alpha E_{-\alpha}})\, b^{-2\lambda_\beta}(e^{\tilde{z}_\beta E_{-\beta}})\, b^{-2\lambda_\gamma}(e^{\tilde{z}_\gamma E_{-\gamma}}).$$

By Lemma 2.3 with $a = 1$ this becomes

$$|c^\lambda_\mathrm{s}(g_\alpha . g_\beta . g_\gamma . \xi_0)|^2 = \mathrm{a}^{-l_\alpha}(\tilde{z}_\alpha)\mathrm{a}^{-l_\beta}(\tilde{z}_\alpha)\mathrm{a}^{-l_\gamma}(\tilde{z}_\gamma).$$

It remains to calculate the $\tilde{z}_\alpha, \tilde{z}_\beta, \tilde{z}_\gamma$ as functions of the coordinates $z_\alpha, z_\beta, z_\gamma$ of

$$\xi := e^{z_\alpha E_\alpha} s_\alpha . e^{z_\beta E_\beta} s_\alpha . e^{z_\gamma E_\gamma} s_\gamma \xi_0 \,.$$

Thus take

$$g_\alpha := e^{z_\alpha E_\alpha} s_\alpha,\; g_\beta := e^{z_\beta E_\beta} s_\alpha,\; g_\gamma := e^{z_\gamma E_\gamma}$$

or equivalently

$$s_\alpha g_\alpha = e^{-z_\alpha E_{-\alpha}}, s_\alpha g_\beta = s_\alpha e^{-z_\beta E_{-\beta}}, s_\gamma g_\gamma = e^{-z_\gamma E_{-\gamma}}.$$

The problem is this. Given $b_\alpha \in B_\alpha$ find $b_\beta \in B_\beta$ from $b_\beta = b(b_\alpha e^{-z_\beta E_{-\beta}})$ in order to determine $\tilde{v}_\gamma = e^{\tilde{z}_\gamma E_{-\gamma}} \in \mathrm{N}_\gamma^-$ from $\tilde{v}_\gamma = v(b_\beta e^{-z_\gamma E_{-\gamma}})$.

At the first step of the process determine b_α from $b_\alpha = b(g_\alpha)$. For $g_\alpha = e^{-z_\alpha E_{-\alpha}}$ the equation becomes

$$b_\alpha = b(e^{-z_\alpha E_{-\alpha}}).$$

By Lemma 2.3 (which is not affected by the minus sign in the exponent)

$$b_\alpha = b(e^{-z_\alpha E_{-\alpha}}) \in \tilde{a}_\alpha \mathrm{N}_\beta,\; \tilde{a}_\alpha = \mathrm{a}^{H_\alpha/2}(\tilde{z}_\alpha) \quad \tilde{z}_\alpha := z_\alpha.$$

At the second step, determine b_β from the equation

$$b_\beta = b(b_\alpha e^{-z_\beta E_{-\beta}}), \quad b_\alpha \in \tilde{a}_\alpha \mathrm{N}_\alpha.$$

$b_\alpha e^{-z_\beta E_{-\beta}}$ is the action of b_α on B_β through the connecting homomorphism, which is trivial on N_α. Hence b_α may be replaced by $\tilde{a}_\alpha$ and same lemma applies:

$$\begin{aligned} &b_\beta = b(\tilde{a}_\alpha e^{-z_\beta E_{-\beta}}) \in \tilde{a}_\beta \mathrm{N}_\beta, \\ &\tilde{a}_\beta := \tilde{a}_\alpha \mathrm{a}^{H_\beta}(\tilde{z}_\beta) = \mathrm{a}^{H_\alpha}(\tilde{z}_\alpha)\mathrm{a}^{H_\beta}(\tilde{z}_\beta), \\ &\tilde{z}_\beta := (\tilde{a}_\alpha)^{-\beta} z_\beta = \mathrm{a}^{-\beta(H_\alpha)/2}(\tilde{z}_\alpha) z_\beta. \end{aligned}$$

At the third step,

$$b_\gamma = b(b_\beta e^{-z_\gamma E_{-\gamma}}), \quad b_\beta \in \tilde{a}_\beta \mathrm{N}_\beta.$$

This is

$$\begin{aligned} &b_\gamma = b(\tilde{a}_\beta e^{-z_\gamma E_{-\gamma}}) \in \tilde{a}_\gamma \mathrm{N}_\gamma, \\ &\tilde{a}_\gamma := \tilde{a}_\beta \mathrm{a}^{H_\gamma}(\tilde{z}_\gamma) = \mathrm{a}^{H_\alpha}(\tilde{z}_\alpha)\mathrm{a}^{H_\beta}(\tilde{z}_\beta)\mathrm{a}^{H_\gamma}(\tilde{z}_\gamma), \\ &\tilde{z}_\gamma := (\tilde{a}_\beta)^{-\gamma} z_\gamma = \mathrm{a}^{-\gamma(H_\alpha)/2}(\tilde{z}_\alpha)\mathrm{a}^{-\gamma(H_\beta)/2}(\tilde{z}_\beta)\mathrm{a}^{-\gamma(H_\beta)/2}(\tilde{z}_\gamma) z_\gamma \end{aligned}$$

These are the desired formulas for these variables. The result is

$$|c_{\mathrm{s}}^\lambda(\xi)|^2 = \prod \mathrm{a}^{-\lambda_i(H_{\alpha_i})}(\tilde{z}_i) = a_\lambda(\xi)^{-1}$$

and

$$\sigma_\lambda = -\frac{1}{2\pi \mathrm{i}} \bar{\partial}\partial |c_{\mathrm{s}}^\lambda|^2 = +\frac{1}{2\pi \mathrm{i}} \bar{\partial}\partial a_\lambda.$$

□

5. Canonical coordinates

It will be convenient to change variables:

$$z_i = e^{\zeta_i} \quad \zeta_i := \tau_i + \sqrt{-1}\phi_i$$
$$\tilde{z}_i = e^{\tilde{\zeta}_i} \quad \tilde{\zeta}_i := \tilde{\tau}_i + \sqrt{-1}\tilde{\phi}_i\,.$$

In these variables the formula for σ_λ reads

$$\sigma_\lambda = \frac{1}{2\pi \mathrm{i}} \sum_{ij} \frac{\partial^2 K_\lambda}{\partial\bar{\zeta}_i \partial\zeta_j} d\bar{\zeta}_i \wedge d\zeta_j = \frac{1}{2\pi} \sum_{ij} \frac{1}{2} \frac{\partial^2 K_\lambda}{\partial\tau_i \partial\tau_j} d\tau_i \wedge d\phi_j,$$
$$K_\lambda = \sum_i l_i \mathrm{K}(\tilde{\tau}_i) \quad \mathrm{K}(\tilde{\tau}) := \log \mathrm{a}(\tilde{\tau}) \quad \mathrm{a}(\tilde{\tau}) := 1 + e^{2\tilde{\tau}}$$

Also introduce

$$J_i := \frac{1}{2}\frac{\partial K_\lambda}{\partial \tau_i}, \quad \mathrm{J}(\tilde{\tau}) := \frac{1}{2}\frac{d\mathrm{K}(\tilde{\tau})}{d\tilde{\tau}} = \frac{e^{2\tilde{\tau}}}{1+e^{2\tilde{\tau}}}$$

The range of J is to be noted: $0 \le \mathrm{J} \le 1$. These formulas have the following consequence.

Theorem 5.1. *The curvature form of $\mathcal{L}_\lambda$ is given by*

$$\sigma_\lambda = \frac{1}{2\pi} \sum_i dJ_i \wedge d\phi_i\,,$$

and the action of $e^H \in T_{\mathrm{s}}$ is given by

$$(J_i, \phi_i) \circ e^H = (J_i, \phi_i + \varpi_i(H)).$$

Define linear combinations $L_1, \cdots, L_\ell$ of the J_i's by

$$L_1 := 0, \quad L_j := \sum_{i:j>i} c_{ji} J_i.$$

Then $J_1, \cdots, J_\ell$ are recursively determined by

$$J_1 = l_1 \mathrm{J}_1, \quad J_j = \mathrm{J}_j (l_j - L_j), \quad [\mathrm{J}_j := \mathrm{J}(\tilde{\tau}_j)]$$

and their range is determined by $0 \le \mathrm{J}_j \le 1$, i.e.

$$0 \le \frac{J_1}{l_1} \le 1, \quad 0 \le \frac{J_j}{l_j - L_j} \le 1$$

with the understanding that a vanishing denominator means a vanishing numerator.

Proof. The formulas for σ_λ and the action of T_{s} in terms of the variables (J_i, ϕ_i) follows from the corresponding formulas in terms of the variables (z_i). The formula for K_λ reads

$$\frac{1}{2} dK_\lambda = \sum_i l_i \mathrm{J}_i d\tilde{\tau}_i, \quad \mathrm{J}(\tilde{\tau}) := \frac{1}{2}\frac{d\mathrm{K}(\tilde{\tau})}{d\tilde{\tau}}.$$

In terms of the τ_i's,

$$\frac{1}{2} dK_\lambda = \sum_i J_i d\tau_i, \quad J_i := \frac{1}{2}\frac{\partial K_\lambda}{\partial \tau_i}.$$

The relations between the $r_i = e^{\tau_i}$ and the $\tilde{r}_i = e^{\tilde{\tau}_i}$ are

$$r_\ell = \tilde{r}_\ell \quad \cdots \quad r_i = \tilde{r}_i \prod_{j:j>i} \mathrm{a}^{-\frac{1}{2}c_{ji}}(\tilde{r}_j).$$

In terms of the new variables these equations say

$$\tau_i = \tilde{\tau}_i - \sum_{j:j>i} \frac{1}{2} c_{ji} \mathrm{K}(\tilde{\tau}_j)$$

With some rearrangements this gives

$$d\tau_i = d\tilde{\tau}_i + \sum_{j:j>i} c_{ji} \mathrm{J}_j d\tilde{\tau}_j$$

with $\mathrm{J}_j := \mathrm{J}(\tilde{\tau}_j) := (1/2) d\mathrm{K}(\tilde{\tau})/\mathrm{d}\tilde{\tau}$. Use these equations to compare the coefficients of the $d\tilde{\tau}_j$'s in the two expressions

$$\frac{1}{2} dK_\lambda = \sum_j l_j \mathrm{J}_j d\tilde{\tau}_j, \quad \frac{1}{2} dK_\lambda = \sum_i J_i d\tau_i \,.$$

Proceeding in the order $d\tilde{\tau}_1, d\tilde{\tau}_2, \cdots$ find

$$l_1 \mathrm{J}_1 = J_1, \quad l_j \mathrm{J}_j = J_j + \sum_{i:j>i} c_{ji} \mathrm{J}_j J_i$$

as required. □

Some comments. (1)The theorem agrees with the results of Grossberg and Karshon [1994], but their 2-form is constructed differently, by a rather special procedure adapted to their setting. (2)If the form σ_λ is everywhere non-degenerate, the preceding theorem says that the canonical coordinates (J_i, ϕ_i) are action-angle variables for action of the compact real form of T_{s} on the symplectic manifold $\mathcal{X}_{\mathrm{s}}, \sigma_\lambda$ [Arnold, 1989]. In this context σ_λ is the object of interest, and indeed as form, not only as cohomology class. (3)In contrast to the construction of the variables $\tilde{r}_\ell, \cdots, \tilde{r}_1$, the inductive construction of $J_1, \cdots, J_\ell$ in terms of $\tilde{\tau}_1, \cdots, \tilde{\tau}_\ell$ proceeds in the same direction as the construction of the $\mathbb{P}^1$-chain. As a consequence, $J_j = \mathrm{J}_j(l_j - \sum_{i:j>i} c_{ji} J_i)$ depends only on the variables $\tilde{\tau}_i$ with $i \le j$ but on all of the variables τ_i.

For a curvature form of a holomorphic line bundle it is positivity which is of primary interest, rather than just non-degeneracy. This is the question to be addressed next.

6. Positivity Questions

Let $c_{ji}^{\pm} := c_{ji}$ if $\pm c_{ji} > 0$ and $= 0$ otherwise. Let $J_{j\,\min}, J_{j\,\max}$ be the min / max of $J_j := \mathrm{J}_j(l_j - \sum_{i:j>i} c_{ji} J_i)$ over $0 \le \mathrm{J}_j \le 1$ for fixed values of $J_{j-1}, \cdots, J_1$, inductively given by

$$J_{j\,\min} = \min\{0,\, l_j - \sum_{i:j>i} c_{ji}^{+} J_{i\,\max} + c_{ji}^{-} J_{i\,\min}\}$$
$$J_{j\,\max} = \max\{0,\, l_j - \sum_{i:j>i} c_{ji}^{+} J_{i\,\min} + c_{ji}^{-} J_{i\,\max}\}$$

$J_{j\,\min/\max} = J_{j\,\min/\max}(\lambda;\, J_{j-1}, \cdots, J_1\,)$ are linear combinations of $J_{j-1}, \cdots, J_1$ depending on λ.

Let $\min J_j$, $\max J_j$ be the min / max of J_j over its complete range, inductively given by

$$\min J_j = \min\{0,\, l_j - \sum_{i:j>i} c^+_{ji} \max J_i + c^-_{ji} \min J_i\}$$
$$\max J_j = \max\{0,\, l_j - \sum_{i:j>i} c^+_{ji} \min J_i + c^-_{ji} \max J_i\}$$

$\min / \max J_j = \min / \max J_j(\lambda)$ depends on λ only.

Theorem 6.1. *Assume $l_1 \neq 0, \cdots, l_\ell \neq 0$. The form $\sigma_{-\lambda}$ is positive if and only if λ satisfies*

$$\min J_1(\lambda) \geq 0 \quad \cdots \quad \min J_\ell(\lambda) \geq 0.$$

If so, the range of (J_i) is the cubic polytope given by the ℓ independent linear inequalities

$$\pi(\mathcal{X}): \quad 0 \leq J_j \leq l_j - L_j, \quad L_j := \sum_{i:j>i} c_{ji} J_i.$$

Remarks. (1) The conditions $\min J_j(\lambda) \geq 0$ are equivalent to $\min J_j(\lambda) = 0$ and amount to ℓ piecewise linear inequalities in λ. They are certainly satisfied if $l_\ell \gg l_{\ell-1} \gg \cdots \gg l_1 > 0$; one only needs to insure that so that the first term in J_j $= \mathrm{J}_j(l_j - \sum_{i:j>i} c_{ji} J_i)$ dominates the rest.

(2)The action variables J_j are determined only up to an additive constant depending on λ by the equation $\sigma_\lambda = \frac{1}{2\pi} \sum_j dJ_i \wedge d\phi_j$. The J_j's used here are normalized so that $J_j|_{z_j=0} = 0$. Without this normalization the positivity condition reads $\min J_j(\lambda) = J_j|_{z_j=0}$. The polytope $\pi(\mathcal{X})$ is still given by $J_{j\,\min} \leq J_j \leq J_{j\,\max}$, provided 0 is replaced by $J_j|_{z_j=0}$ in the definition of $J_{j\,\min/\max}$.

The proof of the theorem needs some preparation. We have

$$J_j = \mathrm{J}_j(l_j - L_j), \quad L_j = \sum_{i:j>i} c_{ji} J_i.$$

L_j depends only on the variables $\tilde\tau_i$ with $j > i$ and

$$\mathrm{J}_i := \mathrm{J}(\tilde\tau_i), \quad \mathrm{K}(\tilde\tau) := \log \mathrm{a}(\tilde\tau), \quad \mathrm{J}(\tilde\tau) := \frac{1}{2} \frac{\mathrm{dK}(\tilde\tau)}{\mathrm{d}\tilde\tau}.$$

Lemma 6.2. *The curvature form of $\mathcal{L}_\lambda$ on $\mathcal{X}_\mathrm{s}$is given by*

$$\sigma_\lambda = \sum_{ji=1}^{\ell} \frac{1}{2} \frac{\partial^2 K_\lambda}{\partial\tau_i \partial\tau_j} \frac{1}{4\pi\mathrm{i}} \frac{d\bar\zeta_i \wedge d\zeta_j}{\bar\zeta_i \zeta_j}\,.$$

Its ℓth exterior power is

$$\sigma^\ell_\lambda = D_\lambda \prod_{j=1}^{\ell} \tilde\sigma_{-\alpha_j/2}, \quad D_\lambda := \prod_j (l_j - L_j), \quad \tilde\sigma_{-\alpha_j/2} := \prod_{j=1}^{\ell} \frac{\mathrm{K}''(\tilde\tau_j)}{2\pi\mathrm{i}} d\tilde\tau_j \wedge d\tilde\phi_j,$$

$\tilde\sigma_{-\alpha_j/2}$ being the curvature form on a copy of $\mathbb{P}^1$ itself with affine coordinate $\tilde\zeta_j = e^{\tilde\tau_j + \sqrt{-1}\tilde\phi_j}$.

Proof. (*Lemma*) Write $\sigma_\lambda = \frac{1}{2\pi\mathrm{i}} \bar\partial\partial K_\lambda$ in terms of the operators

$$\partial = \sum d\zeta_j \frac{\partial}{\partial\zeta_j} \quad \bar\partial = \sum d\bar\zeta_j \frac{\partial}{\partial\bar\zeta_j} \quad [d = \bar\partial + \partial].$$

Use the relations

$$dK_\lambda = 2\sum J_j d\tau_j, \quad d\tau_j = \frac{1}{2}(\partial \log\zeta_j + \bar\partial \log\bar\zeta_j) = \frac{1}{2}(\frac{d\zeta_j}{\zeta_j} + \frac{d\bar\zeta_j}{\bar\zeta_j})$$

to compute

$$\sigma_\lambda = \frac{1}{2\pi \mathrm{i}}\bar\partial\partial K_\lambda = \frac{1}{\pi\mathrm{i}}\sum \bar\partial J_j \wedge \partial\tau_j = \frac{1}{4\pi\mathrm{i}}\sum \frac{\partial J_j}{\partial \tau_i}\frac{d\bar\zeta_i \wedge d\zeta_j}{\bar\zeta_i\zeta_j}.$$

This proves the first assertion. To prove the second one, note that

$$\det\frac{\partial J_j}{\partial\tau_i} = \det(\frac{\partial J_j}{\partial\tilde\tau_l})\det(\frac{\partial\tilde\tau_l}{\partial\tau_i}) = \prod_j \frac{\partial J_j}{\partial\tilde\tau_j}$$

since both determinants are triangular, the second with diagonal entries $= 1$. Since L_j depends only on the variables $\tilde\tau_i$ with $j > i$,

$$\frac{\partial J_j}{\partial\tilde\tau_j} = \frac{\partial((l_j - L_j)\mathrm{J}_j)}{\partial\tilde\tau_j} = (l_j - L_j)\mathrm{J}'(\tilde\tau_j) = \frac{1}{2}(l_j - L_j)\,\mathrm{K}''(\tilde\tau_j).$$

Thus

$$\det\frac{\partial J_j}{\partial\tau_i} = \prod_j (l_j - L_j)\,\frac{1}{2}\,\mathrm{K}''(\tilde\tau_j), \quad \sigma_\lambda^\ell = D_\lambda\prod_{j=1}^{\ell}\frac{\mathrm{K}''(\tilde\tau_j)}{8\pi\mathrm{i}}\frac{d\bar\zeta_j\wedge d\zeta_j}{\bar\zeta_j\zeta_j}.$$

Since $\zeta_j = \tilde\zeta_j$+terms depending only on $\tilde\zeta_{j+1},\cdots,\tilde\zeta_\ell$, the ζ_j can be replaced by $\tilde\zeta_j$ in last formula. In terms of to $\tilde\tau_j, \tilde\phi_j$ this gives

$$\sigma_\lambda^\ell = D_\lambda\prod_{j=1}^{\ell}\tilde\sigma_{-\alpha_j/2}, \quad \tilde\sigma_{-\alpha_j/2} := \prod_{j=1}^{\ell}\frac{\mathrm{K}''(\tilde\tau_j)}{2\pi\mathrm{i}}d\tilde\tau_j\wedge d\tilde\phi_j.$$

This proves the second assertion. □

Proof. (*Theorem*) Generally, a (1,1) form $\frac{1}{2\mathrm{i}}\sum K_{ij}d\bar z_i\wedge dz_j$ is positive iff the matrix (K_{ij}) is positive definite. So the form $\sigma_{-\lambda}$ is positive definite at a point ξ if and only if the matrix $\partial^2 K_{+\lambda}/\partial\tau_i\partial\tau_j$ is, and this happens if and only if all principal minors of this matrix are positive [Gantmacher, 1959, p.307]. Stated in terms of forms rather than matrices, this criterion reads as follows. *A $(1,1)$ form σ is positive at a point if and only if the sequence of exterior powers $\sigma^\ell, \sigma^{\ell-1},\cdots,\sigma$ is positive on a complete flag in the tangent space*. For a form of top degree, like σ^ℓ, positivity means $\sigma^\ell = D\prod\frac{1}{2\mathrm{i}}d\bar z_j\wedge dz_j$ with $D > 0$, by definition.

To argue by induction on ℓ, write the $\mathbb{P}^1$-chain $\mathcal{X}_\mathrm{s}$ as $\mathcal{X}_\mathrm{s} = G_\ell\times_{B_\ell}\mathcal{X}_{\mathrm{s}'}$ where $\mathcal{X}_{\mathrm{s}'}$ is the $\mathbb{P}^1$-chain built from the sequence s' obtained from s by omitting the last letter $i = \ell$. $\mathcal{X}_{\mathrm{s}'}$ is embedded in $\mathcal{X}_\mathrm{s}$ as the fiber of $\mathcal{X}_\mathrm{s}\to G_\ell/B_\ell\approx\mathbb{P}^1$ over the base-point $1B_\ell$. The coordinate z_ℓ on $\mathcal{X}_\mathrm{s}$ is the pull-back of the affine coordinate on centered at the point $s_\ell B$, also denoted z_ℓ. The submanifold $\mathcal{X}_{\mathrm{s}'}$ of $\mathcal{X}_\mathrm{s}$ is then given by the equation $z_\ell\circ s_\ell = 0$. Since $\tilde\zeta_\ell = \zeta_\ell$, the formula for σ_λ^ℓ shows that at such a point

$$\sigma_\lambda^\ell\big|_{z_\ell\circ s_\ell=0} = \left[(l_\ell - L_\ell)\frac{\mathrm{K}''(\tau_\ell)}{8\pi\mathrm{i}}\frac{d\bar\zeta_\ell\wedge d\zeta_\ell}{\bar\zeta_\ell\zeta_\ell}\right]_{z_\ell\circ s_\ell=0}\wedge\sigma_{\lambda'}^{\ell-1}$$

where λ' is obtained by omitting the ℓth component and $\sigma_{\lambda'}$ is the corresponding form on $\mathcal{X}_{\mathrm{s}'}$. The form in brackets is a pull-back of this form G_ℓ/B_ℓ written in the coordinate ζ_ℓ. It is positive if and only if $l_\ell - L_\ell$ is positive. Since every point of

$\mathcal{X}_{\mathrm{s}}$ can be brought to $\mathcal{X}_{\mathrm{s}'}$ by the action of the compact form U_ℓ of G_ℓ, which leaves σ_λ invariant, one finds that $\sigma^\ell_\lambda > 0$ on $\mathcal{X}_{\mathrm{s}}$ if and only if

$$l_\ell - L_\ell > 0 \text{ and } \sigma^{\ell-1}_{\lambda'} > 0 \text{ on } \mathcal{X}_{\mathrm{s}'}.$$

$l_\ell - L_\ell = l_\ell + \cdots$ is independent of $\tilde{\tau}_\ell$ and $J_\ell = \mathrm{J}_\ell(l_\ell - L_\ell)$, $0 \le \mathrm{J}_\ell \le 1$. Hence $l_\ell - L_\ell > 0$ amounts to $J_\ell \ge 0$ and not $\equiv 0$ on $\mathcal{X}_{\mathrm{s}}$. Since $J_\ell = \mathrm{J}_\ell l_\ell + \cdots$ the latter condition is satisfied as soon as $l_\ell \ne 0$, which holds by hypothesis, so that we may replace $l_\ell - L_\ell > 0$ by $J_\ell \ge 0$. Thus $\sigma_{-\lambda} = -\sigma_\lambda$ is positive if and only if $J_1 \ge 0, \cdots, J_\ell \ge 0$ throughout, i.e. $\min J_j(\lambda) = 0$. If so, the range of the $J_j = \mathrm{J}_j(l_j - L_j), 0 \le \mathrm{J}_j \le 1$, is indeed the given by $0 \le J_j \le l_j - L_j$. □

The condition for the positivity of σ_λ will be referred to as the *positivity condition* on λ.

7. Quantization

Projective embedding. Kodaira's theorems assert the following consequences of positivity [Griffiths and Harris, 1978, p.156, 181].

Theorem 7.1. *Assume λ satisfies the positivity condition.*

(1) *Let $\Omega^q(\mathcal{L}_\lambda)$ be the sheaf of holomorphic q-forms with values in $\mathcal{L}_\lambda$.*

$$\mathrm{H}^p(\mathcal{X}_{\mathrm{s}}, \Omega^q(\mathcal{L}_\lambda)) = 0 \text{ for } p + q > \ell.$$

(2) *Let $\mathcal{O}(\mathcal{L}_\lambda)$ be the sheaf of holomorphic sections of $\mathcal{L}_\lambda$. The natural map*

$$\mathcal{X}_{\mathrm{s}} \to \mathbb{P}(\mathcal{H}^*), \mathcal{H} := \mathrm{H}^0(\mathcal{X}_{\mathrm{s}}, \mathcal{O}(\mathcal{L}_{k\lambda})$$

is an embedding for any sufficiently large multiple $k\lambda$ of λ.

Some comments. In general, the natural map $\mathcal{X} \to \mathbb{P}(\mathcal{H}^*)$ into the projective space dual to the space $\mathcal{H} := \mathrm{H}^0(\mathcal{X}, \mathcal{O}(\mathcal{L}))$ of global holomorphic sections is defined as follows. Given $\xi \in \mathcal{X}$, choose any non-zero element $e(\xi)$ in the line $\mathcal{L}(\xi)$. The map $f \mapsto f(\xi)/e(\xi)$ defines a linear functional $e_\xi \in \mathcal{H}^*$, which up to a scalar depends only on ξ and represents the image of ξ in $\mathbb{P}(\mathcal{H}^*)$. The line bundle $\mathcal{L}$ on $\mathcal{X}$ is the pull-back of the hyperplane bundle on the ambient projective space $\mathbb{P}(\mathcal{H}^*)$, so its Chern class is the pull-back of that of the hyperplane bundle. These Chern classes are represented by (normalized) curvature forms $\sigma_{\mathcal{X}}, \sigma_{\mathbb{P}^N}$ for any unitary metrics on these line bundles. As form, however, $\sigma_{\mathcal{X}}$ need not coincide with the pull-back of the Fubini-Study form $\sigma_{\mathbb{P}^N}$; on the contrary, the relation between these forms is generally a delicate question. In particular, the unitary norm on the Hilbert space of sections, which is of interest for applications to representation theory, depends on the form $\sigma_{\mathcal{X}}$ itself.

Some notation. Throughout λ is assumed to satisfy the positivity condition and will remain fixed. It will be dropped as a subscript, as will be the subscript s, e.g. on σ and $\mathcal{X}$. Write T for the image of T_{s} as transformation group on $\mathcal{X}$. Like any complex torus, T has a unique decomposition $T = T_{\mathrm{U}} T_{\mathrm{P}}$ into a 'unitary' real form $T_{\mathrm{U}} \approx \prod \mathrm{U}(1)$ and a 'positive' real form $T_{\mathrm{P}} \approx \prod \mathrm{GL}(1,\mathbb{R})_{\text{positive}}$. The weights ϖ_i by which T acts on the coordinates z_i via $z_i \circ h = h^{\varpi_i} z_i$ form a $\mathbb{Z}$–basis for a lattice $\mathfrak{t}_{\mathrm{P}}(\mathbb{Z})$ in the real dual of $\mathfrak{t}^*_{\mathrm{P}}$ of the of the Lie algebra of T_{P}. Let N_i be the dual basis for the dual lattice $\mathfrak{t}_{\mathrm{P}}(\mathbb{Z})$ in $\mathfrak{t}_{\mathrm{P}}(\mathcal{X}) \subset \mathfrak{t}_{\mathrm{P}}$, defined by $\langle \varpi_i, N_j \rangle = \delta_{ij}$. The N_i form an $\mathbb{R}$-basis for $\mathfrak{t}_{\mathrm{P}}$ and the $\mathrm{i}N_i$ for $\mathfrak{t}_{\mathrm{U}}$. The imaginary unit i:= $\sqrt{-1}$ will be written explicitly whenever $\mathfrak{t}_{\mathrm{U}}$ is concerned: $\mathfrak{t}_{\mathrm{P}}$ is taken as "the" real form of $\mathfrak{t}$

and $\mathfrak{t} = \mathfrak{t}_{\mathrm{P}} + \mathfrak{t}_{\mathrm{U}}$ as the decomposition of $\mathfrak{t}$ into "real" and "imaginary" parts. For example, write

$$\mathrm{i}H \in \mathfrak{t}_{\mathrm{U}} : H = \sum_i \epsilon_i N_i, \epsilon_i \in \mathbb{R}, \quad \mathrm{i}\eta \in \mathfrak{t}_{\mathrm{U}}^* : \eta = \sum_i n_i \varpi_i \, , \, n_i \in \mathbb{R}.$$

Thus i functions as map $\mathfrak{t}_{\mathrm{P}} \to \mathfrak{t}_{\mathrm{U}}$. The pairing of $H \in \mathfrak{t}_{\mathrm{P}}$ and $\eta \in \mathfrak{t}_{\mathrm{P}}^*$ is written as $\langle \eta, H\rangle$ or $\langle H, \eta \rangle$.

The moment set. Any element $\mathrm{i}H = \mathrm{i}\sum \epsilon_i N_i$ of $\mathfrak{t}_{\mathrm{U}}$ has $J_{\mathrm{i}H}(\xi) := 2\pi \sum \epsilon_i J_i(\xi)$ as a Hamiltonian function, in the sense that $dJ_{\mathrm{i}H} = -\iota(\mathrm{i}H)\sigma$, the inner product of σ with the vector field on $\mathcal{X}$ induced by $-\mathrm{i}H \in \mathfrak{t}_{\mathrm{U}}$. This follows from the formulas $\sigma = \frac{1}{2\pi}\sum dJ_i \wedge d\phi_i$, $\iota(\mathrm{i}N_i)dJ_j = 0$, $\iota(\mathrm{i}N_i)d\phi_j = \delta_{ij}$. In particular the Hamiltonian function for $\mathrm{i}N_i$ is the action variable $J_i(\xi)$. Since $J_{\mathrm{i}H}(\xi)$ depends linearly on $\mathrm{i}H$, it defines a map $\mathcal{X} \to \mathfrak{t}_{\mathrm{U}}^*$, the *moment map* of the T_{U} action on $\mathcal{X}$. Transferred to $\mathfrak{t}_{\mathrm{P}}^*$ it becomes a map $\pi : \mathcal{X} \to \mathfrak{t}_{\mathrm{P}}^*$, defined by $\langle \pi(\xi), H\rangle := J_{\mathrm{i}H}(\xi)$ i.e. $\pi(\xi) = \sum J_i(\xi)\varpi_i$. The image $\pi(\mathcal{X})$ of π in $\mathfrak{t}_{\mathrm{P}}^*$ can evidently be identified with the range of the J_i and will be called the *moment set*.

The weight set. Let $\mathcal{H}$ be the space of holomorphic sections of $\mathcal{L}$, a finite-dimensional space with a positive definite norm defined by the formula

$$\|f\|^2 := \int_{\mathcal{X}} |f|^2 \, \sigma^\ell.$$

There is a natural representation of the complex torus on $\mathcal{H}$, defined by

$$(U_h f)(\xi) := f(h^{-1}\xi),$$

which is unitary on the real torus T_{U}.

Any $\mathrm{i}\eta \in \mathfrak{t}_{\mathrm{U}}^*(\mathbb{Z})$ defines a Laurent monomial $z^\eta := z_1^{n_1} \cdots z_\ell^{n_\ell}$ in the coordinates $z_i = e^{\tau_i + \mathrm{i}\phi_i}$ on $\mathcal{X}$ and hence a holomorphic section $f_\eta = z^\eta c^\lambda$ on the coordinate domain. (f_η transforms by $\tilde{\eta} := \eta + \tilde{\lambda}$ under T_{U} since c^λ transforms by $\tilde{\lambda}$.) The *weight set* Π is the set of weights $\eta \in \mathrm{i}^{-1}\mathfrak{t}_{\mathrm{U}}^*(\mathbb{Z})$ for which f_η extends to a holomorphic section on all of $\mathcal{X}$, a condition which may be reformulated in other ways.

Lemma 7.2. *Let $f = z^\eta c^\lambda$. The following conditions are equivalent.*
(1) *$f(\xi)$ extends to a global holomorphic section on $\mathcal{X}$.*
(2)*$|f(\xi)|$ is bounded on $\mathcal{X}$.*
(3) *$|f(\xi)|^2\sigma^\ell$ is integrable over $\mathcal{X}$.*

Proof. It suffices to prove the equivalence locally, i.e. with $\mathcal{X}$ replaced by a neighbourhood of each of its points. Since the coordinates are meromorphic functions on $\mathcal{X}$, f is a meromorphic section on $\mathcal{X}$, represented locally by a meromorphic function, regular of on coordinate domain, and for such a function the local conditions are indeed equivalent. □

Theorem 7.3. *The weight set $\Pi \subset \mathfrak{t}_{\mathrm{P}}^*$ is $\Pi = \pi(\mathcal{X}) \cap \mathrm{i}^{-1}\mathfrak{t}_{\mathrm{U}}^*(\mathbb{Z})$, i.e. $f_\eta = z^\eta c^\lambda$ extends to a global holomorphic section on all of $\mathcal{X}$ if and only if η belongs to the image $\pi(\mathcal{X})$ of the moment map. These f_η, $\eta \in \Pi$, form a basis for $\mathcal{H}$.*

Proof. The coordinates $z_i = e^{\tau_i + \mathrm{i}\phi_i}$ on $\mathcal{X}$ provide a bijection

$$T_{\mathrm{U}}\backslash\mathcal{X} \leftrightarrow \mathfrak{t}_{\mathrm{P}}, \quad T_{\mathrm{U}}\xi \mapsto H := \sum \tau_i(\xi) N_i,$$

defined on the domain where all coordinates are non-zero. Equivalently, $\xi = e^H \xi_1$ where ξ_1 has coordinates $z_1 = \cdots = z_\ell = 1$. The pointwise square norm $a(\xi) =$

$|c^\lambda(\xi)|^2$ is really a function on $T_{\mathrm{U}}\backslash\mathcal{X}$. Use the bijection $T_{\mathrm{U}}\xi \mapsto H$ to write the equation $K(\xi) = \log a(\xi)$ as $K(H) = \log a(H)$ in terms of $H \in \mathfrak{t}_{\mathrm{P}}$.

By the lemma, f_η extends iff its pointwise square-norm is bounded. This square norm is

$$|f_\eta|(\xi)^2 = |z^\eta c^\lambda(\xi)|^2 = |z|^{2\eta} e^{-K(\xi)} = e^{2\langle\eta,H\rangle - K(H)}$$

where $H = \sum_i \epsilon_i N_i$, $\epsilon_i = \tau_i(\xi)$. Thus the condition is precisely that $2\langle\eta, H\rangle - K(H)$ be bounded from above as function of H. Since the curvature form

$$\sigma = \sum_{ji=1}^{\ell} \frac{1}{2}\frac{\partial^2 K(H)}{\partial\tau_i\partial\tau_j}\frac{1}{4\pi\mathrm{i}}\frac{d\bar\zeta_i \wedge d\zeta_j}{\bar\zeta_i\zeta_j}\,.$$

is positive definite, so is the Hessian matrix of $K(H)$. Hence $K(H)$ *is strictly convex* as function of $H \in \mathfrak{t}_{\mathrm{P}}$ [Rockafellar, 1970, p.27]. For a smooth convex function $K(H)$, $2\langle\eta, H\rangle - K(H)$ achieves a finite maximum iff its partials vanish at some point H_o [loc. cit., p.258], and this amounts to $2n_i = \partial K/\partial\tau_i = 2J_i$ at H_o, i.e. $\eta \in \pi(\mathcal{X})$. The elements $f_\eta, \eta \in \Pi$, are precisely the weight vectors for the representation of T_{U} on $\mathcal{H}$, hence form a basis. $\square$

Corollary 7.4. *The function* $Z(\mathrm{i}H) := \mathrm{trace}(e^{\mathrm{i}H} \mid \mathcal{H})$ *of* $e^{\mathrm{i}H} \in T_{\mathrm{U}}$ *is given by the formula*

$$Z(\mathrm{i}H) = \sum_{\eta\in\Pi} e^{\mathrm{i}\eta(H)}\,.$$

A comment. The determination of the weight set by convex calculus, as in the proof of the theorem, is as general as it is simple. Useful tools are available, especially the apparatus of conjugate functions. By definition, the *conjugate* of a convex function $K(H)$ is $K^{\mathrm{c}}(\eta) := \max\{2\langle\eta, H\rangle - K(H) \mid H \in \mathfrak{t}_{\mathrm{P}}\}$ as function on the dual space $\mathfrak{t}_{\mathrm{P}}^*$. (In this context max $= \infty$ is allowed, by convention.) In the language of convex analysis, the weight set could be described as the set of integral points in the domain of the conjugate function $K^{\mathrm{c}}(\eta), \eta \in \mathfrak{t}_{\mathrm{P}}^*$ of the Kähler potential $K(H), H \in \mathfrak{t}_{\mathrm{P}}$. To an analyst convex conjugates will be familiar as asymptotic phase functions according to the principle of stationary phase, [Guillemin and Sternberg, 1977, p.397].

Partition functions. Whether the function $Z(\mathrm{i}H)$ is to be called *(quantum) partition function* or *character* is a matter of taste and tradition. The prejudice displayed here is adopted for this occasion only, because of a curious relation between toric geometry and statistical mechanics, as will now be explained .

Fix $\mathrm{i}H \in \mathfrak{t}_{\mathrm{U}}$ and let $J := J_{\mathrm{i}H}$ be its Hamiltonian function. The classical density associated to this Hamiltonian is the function on $\mathcal{X}$ by the formula

$$\rho_{\text{classical}}(\xi) := \frac{1}{Z_{\text{classical}}}\frac{e^{-\beta J(\xi)}}{\ell!}, \qquad \int_{\mathcal{X}} \rho_{\text{classical}}\,\sigma^\ell = 1$$

[Honerkamp 1998, p.28]. The denominator $Z_{\text{classical}}$ is a normalizing factor, depending only on H:

$$Z_{\text{classical}} := \int_{\mathcal{X}} e^{-\beta J}\frac{\sigma^\ell}{\ell!}.$$

It can be written in the form

$$Z_{\text{classical}} := \int_{\mathcal{X}} e^{-\beta J + \sigma}$$

the exponential being taken in the exterior algebra with the understanding that the integral picks out the component of the relevant degree $2\ell = \dim_{\mathbb{R}} \mathcal{X}$.

On the other hand, the quantum density is the density in the spectral resolution of $e^{-\beta \mathrm{i} H}$ as operator on $\mathcal{H}$:

$$\rho_{\text{quantum}}(\eta) := \frac{m(\eta)}{Z_{\text{quantum}}}\, e^{-\beta \mathrm{i}\eta}, \quad \sum_{\eta} \rho_{\text{quantum}}(\eta) = 1,$$

$m(\eta)$ being the multiplicity of the $\mathrm{i}\eta$ as eigenvalue of $\mathrm{i}H$ on $\mathcal{H}$. The normalizing factor Z_{quantum} is therefore

$$Z_{\text{quantum}} = \sum_{\eta} m(\eta) e^{-\beta \mathrm{i}\eta} = \mathrm{Tr}(e^{-\beta \mathrm{i} H} \mid \mathcal{H}).$$

[Honerkamp, 1998, p.204]. To bring out the dependence on $\mathrm{i}H \in \mathfrak{t}_{\mathrm{U}}$ write $\rho(\mathrm{i}H; \eta)$ and $Z(\mathrm{i}H)$, if necessary.

The fundamental relation between the classical and quantum partition function is **Kirillov's Formula** [Berline, Getzler, and Vergne, p.250]. In the present situation it states that for $\mathrm{i}H$ in a neighbourhood of zero in $\mathfrak{t}_{\mathrm{U}}$, on which the exponential map is one-to-one, one has

$$\int_{\mathcal{X}} e^{J_{\mathrm{i}H} + \sigma} \widehat{A}(\mathrm{i}H) = \sum_{\eta} m(\eta) e^{\mathrm{i}\langle \eta, H\rangle}.$$

The sum on the right is the quantum partition function and the integral on the left is the classical partition function, except for the extra factor $\widehat{A}(\mathrm{i}H)$ under the integral, a characteristic class involving the Riemann curvature of $\mathcal{X}$, whose definition can be found loc. cit. (Its appearance here suggests that the classical definition of the partition function should be modified so as to incorporate this factor, a modification which is irrelevant in a flat phase space.) That $\mathrm{i}H$ has to remain in neighbourhood of zero is a serious restriction.

Returning to the toric setting, assume that Kodaira's map $\psi : \mathcal{X} \to \mathbb{P}(\mathcal{H}^*), \xi \mapsto \psi_\xi$,is an embedding. Let $\mathcal{X}'$ be the image of $\mathcal{X}$ and T' the image of T as transformation group of $\mathcal{X}'$. Let $\mathcal{L}'$ be the line bundle on $\mathcal{X}'$ induced by the hyperplane bundle on $\mathcal{H}^*$. Let σ' be the curvature from for the unitary metric on $\mathcal{L}'$ induced by any T_{U}-invariant unitary metric on $\mathcal{H}^*$, for example the metric defined above. Let $J'_{\mathrm{i}H'}(\xi)$ be a Hamiltonian function for $\mathrm{i}H' \in \mathfrak{t}'_{\mathrm{U}}$ acting on $(\mathcal{X}', \sigma')$, i.e. $dJ'_{\mathrm{i}H'} = -\iota(\mathrm{i}H')\sigma'$. This equation determines $J'_{\mathrm{i}H'}(\xi)$ up to a constant depending linearly on $\mathrm{i}H'$. Fix this constant as follows. Let ξ_1 be a base-point for $\mathcal{X}$ in the open T-orbit, say the point with all coordinates z_i equal to 1. Let ξ'_1 be it image in $\mathcal{X}'$, and require that $J'_{\mathrm{i}H'}(\xi'_1) = J_{\mathrm{i}H}(\xi_1)$ if $\mathrm{i}H' \in \mathfrak{t}'_{\mathrm{U}}$ is the image of $\mathrm{i}H \in \mathfrak{t}_{\mathrm{U}}$. Thus we have another set of data $\mathcal{X}', \sigma' \cdots$ like $\mathcal{X}, \sigma \cdots$.

Lemma 7.5. *Under the identification* $\mathfrak{t}' = \mathfrak{t}$,

$$Z'_{\text{classical}}(\mathrm{i}H) = Z_{\text{classical}}(\mathrm{i}H), \quad \mathrm{Z}'_{\text{quantum}}(\mathrm{iH}) = \mathrm{Z}_{\text{quantum}}(\mathrm{iH}).$$

Proof. The equality of the quantum partition functions is obvious, since the line bundles $\mathcal{L}$ and $\mathcal{L}'$ correspond under $\mathcal{X} = \mathcal{X}'$. The equality of the classical partition functions, however, is not, since the forms σ and σ' need not coincide. What is obvious is that their cohomology classes coincide, since both represent the Chern class of $\mathcal{L}' = \mathcal{L}$. Thus $\sigma' = \sigma + d\varphi$ for some smooth 1-form φ, which may be chosen T_{U}-invariant like σ and σ'. Then $d \circ \iota(\mathrm{i}H)\varphi = -\iota(\mathrm{i}H) \circ d\varphi$ for all $\mathrm{i}H \in \mathfrak{t}_{\mathrm{U}}$ and

(suppressing iH) the equations $dJ' + \iota\sigma' = 0$, $dJ + \iota\sigma = 0$, give $dJ' = dJ + d\iota\varphi$, i,e $J' = J + \iota\varphi + C$. The normalization $J'(\xi_1) = J(\xi_1)$ implies $C = 0$, so $J' + \sigma' = J + \sigma + (d+\iota)\varphi$. The rest of the proof uses three observations: (1)$(d+\iota)(J+\sigma) = 0$, (2)$(d+\iota)$ is a derivation, and (3)$(d+\iota)\Phi = d\Phi$+forms of lower degree. Armed with these one computes

$$\begin{aligned}\int_{\mathcal{X}} e^{J+\sigma} e^{(d+\iota)\varphi} &= \int_{\mathcal{X}} (e^{J+\sigma} + e^{J+\sigma}(d+\iota)\Phi)) \quad [e^{(d+\iota)\varphi} =: 1 + (d+\iota)\Phi] \\ &= \int_{\mathcal{X}} (e^{J+\sigma} + (d+\iota)(e^{J+\sigma}\Phi)) \quad [\text{ by (1) and (2) }] \\ &= \int_{\mathcal{X}} e^{J+\sigma} + 0 + 0 \quad [\text{ by Stokes and (3) }]\end{aligned}$$

This equation says $Z'_{\text{classical}} = Z_{\text{classical}}$ except for the factor β, which may be absorbed into J and J'. □

The image of the measure $|\sigma^\ell| = |\prod \frac{1}{2\pi} dJ_i \wedge d\phi_i|$ under $\eta = \pi(\xi)$ i.e. $n_i = J_i(\xi)$ is evidently $\rho(\eta)d\eta$, the Haar measure $d\eta = \prod dn_i$ multiplied by the indicator function $\rho(\eta)$concentrated on $\pi(\mathcal{X})$. After an integration over the angle variables the partition function becomes

$$Z_{\text{classical}}(\mathrm{i}H) = \int_{\eta\in\pi(\mathcal{X})} e^{-\beta\langle\eta,H\rangle} d\eta,$$

the Laplace transform of $\rho(\eta)$. The lemma therefore implies that the moment sets for $\mathcal{X}'$ and $\mathcal{X}$ coincide, i.e. $\pi'(\mathcal{X}) = \pi(\mathcal{X})$ under the identification $\mathfrak{t}' = \mathfrak{t}$.

The main point is the topological nature of classical partition function: $Z_{\text{classical}}(H)$ *depends only on the Chern class of the line bundle.* (This phenomenon is evident in the setting of Kirillov's formula in the form presented loc. cit, where the lemma is understood in terms of equivariant cohomology. The calculation of Fourier transforms like one above was one of the first applications of that theory [Berline and Vergne, 1983].) Replacing σ by σ' allows one to apply the theory of toric varieties, which provides information to the moment map and the weight set. There is in particular an intriguing description of the moment set in terms of the maximum entropy principle of statistical mechanics, due to C. Lee [1990, p.13, Ewald 1996, p.298].

The path integral formula. The unitary representation of T_{U} on $\mathcal{H}$ is a quantization of the symplectic manifold $\mathcal{X}, \sigma$ associated to the totally complex, positive polarization provided by its complex structure [Woodhouse, 1991]; the wave functions are the holomorphic sections $f(\xi)$ which make up $\mathcal{H}$. The quantum evolution operators are those representing T_{U}. On the other hand, the canonical variables $(J, \phi) := (J_i, \phi_i)$ provide a natural real polarization, at least on their domain of definition, whose wave functions are functions of ϕ. (It natural to choose ϕ rather than J, because (J, ϕ) has an interpretation as coordinates on the cotangent bundle of T with ϕ as coordinate on the base.) We now leave the realm of geometric quantization for that of path integral quantization. The evolution operator generated by a Hamiltonian function $H(J, \phi)$ is then supposed to be an integral operator on wave functions $\psi(\phi)$,

$$U_H\psi(\phi'') = \int K_H(\phi'', \phi')\psi(\phi')d\phi',$$

whose kernel is given by a path integral, formally written as

$$K_H(\phi'',\phi') := \int e^{\frac{\mathrm{i}}{\hbar}S_H[J(t),\phi(t)]}.$$

$S_H[J(t),\phi(t)]$ is the action along a path $(J(t),\phi(t) \mid t' \leq t \leq t'')$, defined by

$$S_H[J(t),\phi(t)] := \int_{t'}^{t''} [Jd\phi - H(J,\phi)dt].$$

The path integral for $K_H(\phi'',\phi')$ is supposed to be taken over all such path with $\phi(t') = \phi', \phi(t'') = \phi''$. The quantum partition function $Z(\mathrm{i}H)$, now written without subscript, is the trace of U_H, obtained from $K_H(\phi'',\phi')$ by setting $\phi'' = \phi'$ and integrating. The difficulties encountered when trying to define path integrals as true integrals are well known and need not be discussed here. Instead, I shall give a recipe for evaluating $Z(\mathrm{i}H)$, which can be taken as a definition of this path integral in the special case we are concerned with, i.e. for a $\mathbb{P}^1$-chain $\mathcal{X}$, equipped with its curvature form σ and canonical coordinates (J,ϕ), and for Hamiltonians which are linear functions $H = H(J)$ of the J's only.

Consider first the case of $\mathbb{P}^1$ itself. The coordinates are (J,ϕ) with J ranging over an interval $\pi(\mathcal{X}) : J_{\min} \leq J \leq J_{\max}$ and ϕ over $\mathbb{R}/2\pi\mathbb{Z}$. The Hamiltonian is $H(J) := HJ$ for some constant also denoted H. *Define* the path integral

$$Z(\mathrm{i}H) = \int_{\phi(0)\equiv\phi(1)} e^{\mathrm{i}S_H[J(t),\phi(t)]}$$

by the following recipe.

(1) Fix N and replace $S_H[J(t),\phi(t)]$ by the sum

$$S^{(N)} := [J\phi]_0^1 - \sum_{i=0}^{N-1} \phi_i \Delta J_i + HJ_i\Delta t,$$
$$\Delta J_i := J_{i+1} - J_i, \quad \Delta t := 1/N.$$

(2) Fix $n \in \mathbb{Z}$ and integrate over all sequences $(J_0\phi_0, \cdots, J_N\phi_N)$

$$J_{\min} \leq J_i \leq J_{\max}, \quad -\infty < \phi_i < \infty, \quad \phi_N - \phi_0 = 2\pi n.$$

(3) Sum over $n \in \mathbb{Z}$.

(4) Take the limit $N \to \infty$.

Written as a single formula (not recommended), this amounts to the definition

$$Z(\mathrm{i}H) := \lim_{N\to\infty} \sum_{n=0}^{\infty} \int_{J_{\min}}^{J_{\max}} dJ_N \int_{-\infty}^{\infty} d\phi_N \cdots \int_{J_{\min}}^{J_{\max}} dJ_1 \int_{-\infty}^{\infty} d\phi_1 \times$$
$$\times\, \delta(\phi_N - \phi_0 - 2\pi n) e^{i(J_N\phi_N - J_0\phi_0) - i\sum_{i=1}^{N-1}\phi_i(J_i - J_{i-1}) + HJ_i(\frac{1}{N})}.$$

The integrals in this formula must be interpreted in the sense of generalized functions, i.e. as limits of integrals against compactly supported test functions tending to the constant function $\equiv 1$. The integrals over $J_{\max} \leq J \leq J_{\min}$, in particular, are defined as limits against continuous compactly supported functions identically $\equiv 1$ on this closed interval. In (1), an integration by parts has been applied to the integral for S_H, replacing the form $Jd\phi$ by $-\phi dJ$ before replacing $J(t),\phi(t)$ by $(J_1\phi_1, \cdots, J_N\phi_N)$. In (2), the variables ϕ_i replacing $\phi(t)$ are allowed to vary from $-\infty$ to $+\infty$, while the endpoint condition $\phi(0) \equiv \phi(1) \bmod 2\pi$ is interpreted

as the *sum* over all such paths and *not* by requiring that the angle itself return to its initial value. This is an essential feature.

In the case of a general toric $\mathbb{P}^1$-chain the coordinate pair (J, ϕ) is replaced by ℓ coordinate pairs (J_i, ϕ_i) and the interval $\pi(\mathcal{X}) = \{J_{\max} \leq J \leq J_{\min}\}$ by the cubic polytope $\pi(\mathcal{X}) = \{J_{i\,\min} \leq J_i \leq J_{i\,\max}\}$. The path integral $Z(\mathrm{i}H)$ is then defined as an ℓ-fold integral, each of which is given by the recipe above.

Theorem 7.6. *The path integral for $Z(\mathrm{i}H)$ exists and is given by*

$$Z(\mathrm{i}H) = \sum_{\eta \in \Pi} e^{\mathrm{i}\langle \eta, H\rangle} .$$

Proof. Follow the steps in the definition:

(1) Fix N. As indicated let,

$$S_H = \int_0^1 (J d\phi - HJdt) = [J\phi]_0^1 - \int_0^1 (\phi dJ + HJdt)$$
$$S_H^{(N)} := [J\phi]_0^1 - \sum_{k=0}^{N-1} [\phi_k(J_{k+1} - J_k) + \frac{1}{N} HJ_k]$$

(2) Fix n. For fixed k, the part of the integral involving $(J_k \phi_k)$ is of the form

$$\int_{J_{\min}}^{J_{\max}} dJ_k \int_{-\infty}^{\infty} d\phi_k \, e^{-\mathrm{i}\phi_k(J_{k+1}-J_k) - \frac{\mathrm{i}}{N} HJ_k}$$

Use the Fourier inversion formula on the real line in the form

$$\int_{-\infty}^{\infty} \frac{d\phi_k}{2\pi} \, e^{-\mathrm{i}\phi_k(J_{k+1}-J_k)} = \delta(J_{k+1} - J_k),$$
$$\int_{-\infty}^{\infty} dJ_k \, f(J_k)\delta(J_k - J_{k-1}) = f(J_{k-1}).$$

This gives

$$Z(\mathrm{i}H)^{(N,n)} := \int_{J_{\min}}^{J_{\max}} dJ_N \int_{J_{\min}}^{J_{\max}} dJ_0 \; e^{i(J_N\phi_N - J_0\phi_0) - iHJ_0} \; \delta(J_N - J_0)$$
$$= \int_{J_{\min}}^{J_{\max}} d\eta \, e^{-\mathrm{i}\eta(\phi_N - \phi_0 + H)}$$

(3)By Poisson summation,

$$\sum_{n \in \mathbb{Z}} Z(\mathrm{i}H)^{(N,n)} = \sum_{n \in \mathbb{Z}} \int_{J_{\min}}^{J_{\max}} d\eta \, e^{-\mathrm{i}\eta(2\pi n + H)} = \sum_{\eta \in \Pi} e^{\mathrm{i}\eta H}.$$

(4)The result of (3) is independent of N; the limit $N \to \infty$ is trivial.

The proof has been formulated so that it remains valid without change in the setting of $\mathbb{P}^1$-chains. □

Some comments. (1) The appearance of the delta function $\delta(J_N - J_0)$ in step (2) may be interpreted to mean that the path integral for $Z(\mathrm{i}H)$ is automatically concentrated on paths $(J(t), \phi(t))$ closing in J as well as in ϕ, which is not part of the input.

(2) The theorem gives $Z(\mathrm{i}H)$ again as a sum over the set of integral points in the range of the action variables (J_i). Since the proof is nothing but Poisson's

summation formula applied to the definition of the path integral, as specified in the recipe, the canonical coordinates themselves remain as the essential ingredient. The applicability of the path integral in many other situations seems to indicate that it does capture some basic feature, and not only in a formal way.

(3) In a more philosophical vein, it appears that the unitary-positive splitting $T = T_{\mathrm{U}}T_{\mathrm{P}}$ is reflected in the statistical mechanics vs. quantum mechanics setting of the two formulas for the partition functions. The odd nature of this juxtaposition is an old puzzle [Feynman and Hibbs, 1965, p.296].

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